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INSTABILITY ANALYSIS OF NONLINEAR NEUTRAL DIFFERENTIAL DIFFERENCE SYSTEMS WITH INFINITE DELAYS [1]

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Abstract. In this paper, we consider the instability of a class of neutral nonlinear differential difference systems with infinite delays. A practical sufficient criterion for instability is presented by using the method of Liapunov functions and a nonlinear differential difference inequality.

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1. Introduction

One of the most useful techniques in stability theory for ordinary differential equations and differential difference equations is the method of differential inequalities or so called the comparison method. The main idea of this technique is to determine the stability properties of a higher dimensional equation from those of a low-dimensional equation which is usually called a comparison system, through the appropriate choice of a group of Liapunonv functions or Liapunov functionals (for example, see [17]). In our recent paper [20], a class of rather general nonlinear differential difference inequality with infinite delays was established, and at the same time, this inequality was applied to the instability analysis of *retarded* nonlinear differential difference large scale systems. The purpose of this paper is to extend the inequality analysis technique developed in [20], together with the method of Liapunov functions, to the instability analysis of a class of nonlinear *neutral* differential difference systems with infinite delays.

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As usual, let R^n represent n dimensional real Euclidean space. For any vector $x \in R^n$, $x \geq 0 (> 0)$ means that all elements of x are nonnegative (positive), respectively. Let R_+^n denote the set $\{x | x \in R^n, x \geq 0\}$. Conventionally, we use R and R_+ to denote R^1 and R_+^1, respectively. The notation $a \leq +\infty$ (or $a \geq -\infty$) means that a is a real constant or $+\infty$ (or a real constant or $-\infty$), respectively. For any $b \in R_+$, the notation $[0, b)^n$ denotes the product of n intervals $[0, b)$, i.e., $[0, b) \times ... \times [0, b)$.

The following definitions and lemma follow from [3] and [20], which we require for this paper.

Definition 1. [3] An $n \times n$ real constant matrix $C = (c_{ij})_{n \times n}$ with $c_{ij} \leq 0 (i \neq j,\ i, j = 1, 2, ..., n)$ is said to be an M-matrix, if there is a vector $v > 0$ such that $Cv > 0$ or $C^T v > 0$.

Some other equivalent conditions for an M-matrix can be found in [3].

Definition 2. [20] Let D_+^n be an open subset of R_+^n with $x = 0 \in D_+^n$. The continuous function

$$F(x, y, z) = (f_1(x, y, z), ..., f_n(x, y, z))^T : D_+^n \times D_+^n \times D_+^n \to R^n$$

is said to have *Property* (LM), if $f_i(x, y, z) = f_i(x_1, ..., x_n; y_1, ..., y_n; z_1, ..., z_n)$ is nondecreasing with respect to argument x_i and nonincreasing with respect to arguments $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n; y_1, ..., y_n; z_1, ..., z_n$; and there exists a group of positive constants $d_1, ..., d_n$ such that for $0 < u \leq \delta \leq +\infty$,

$$f_i(d_1 u, ..., d_n u; d_1 u, ..., d_n u; d_1 u, ..., d_n u) \equiv \bar{f}_i(u) > 0, \quad \bar{f}_i(0) = 0, \qquad (1)$$

for $i = 1, 2, ..., n$. If, in addition, $D_+^n = R_+^n$ and $\delta = +\infty$, then, function $F(x, y, z)$ is said to have *Property* (M).

Remark 1. The functions with *Property* (LM) or *Property* (M) and the well known M-functions (see [1,22,27]) are natural nonlinear generalizations of an M-matrix.

The following nonlinear differential difference inequality is a simple generalization of the inequality in [20] and will play an important role in instability analysis of neutral nonlinear differential difference systems in the present paper.

Let $p(t) = \text{col}(p_1(t), ..., p_n(t)) : R \to R_+^n$ is a continuous function which satisfies the following nonlinear differential difference inequality for $t \geq t_0 \geq 0$ and $p_j(s) \leq q$ $(s \leq t,\ 0 < q \leq +\infty;\ j = 1, 2, ..., n.)$,

$$k_i D^+ p_i(t) \;\;\geq\;\; r_i(t) b_i(p_i(t)) f_i(p_1(t), ..., p_n(t); \bar{p}_1(t), ..., \bar{p}_n(t);$$

$$\sum_{k=1}^{m}\int_{\theta}^{t}A_{i1}^{(k)}(t,u)R_{i1}^{(k)}(p_1(u))du,...,$$
$$\sum_{k=1}^{m}\int_{\theta}^{t}A_{in}^{(k)}(t,u)R_{in}^{(k)}(p_n(u))du),\quad i=1,2,...,n, \tag{2}$$

where $D^+p_i(t)$ denotes Dini right-hand upper derivative of $p_i(t)$ at the time t,

$$\bar{p}_i(t)=\sup_{-\Delta(t)\le s\le 0}p_i(t+s),$$

$-\infty\le\theta\le 0$; k_i is a nonnegative constant with $k_1+...+k_n>0$; m is a positive integer; $r_i(t):[t_0,+\infty)\to R_+$, $b_i(u):[0,\sigma_0)\to R_+$, $f_i(x_1,...,x_n;y_1,...,y_n;z_1,...,z_n):[0,\sigma_1)^n\times[0,\sigma_1)^n\times[0,\sigma_1)^n\to R$, $A_{ij}^{(k)}(t,u):[t_0,+\infty)\times R\to R_+$, $R_{ij}^{(k)}(u):[0,\sigma_2)\to R_+$ and $\Delta(t):[t_0,+\infty)\to R_+$ are continuous functions satisfying the following conditions for all $t\ge t_0$ and any $s>0$,

(i) $t-\Delta(t)\to+\infty$ as $t\to+\infty$;

(ii) $r_i(t)>0$, $\int_{t_0}^{+\infty}r_i(t)dt=+\infty$;

(iii) $R_{ij}^{(k)}(u)$ is nondecreasing, $R_{ij}^{(k)}(0)=0$ and $b_i(u)>0$ $(0<u<\sigma_0)$;

(iv) $\int_\theta^t A_{ij}^{(k)}(t,u)du\le s_{ij}^{(k)}=\text{const.}$, $\lim_{t\to+\infty}\int_\theta^s A_{ij}^{(k)}(t,u)du=0$,

where $0<\sigma_l\le+\infty$, $l=0,1,2$, $i,j=1,2,...,n$ and $k=1,2,...,m$.

Lemma 1. *Assume that $(i)-(iv)$ hold, and*

(v) *the function*

$$F(x,y,z)\equiv(f_1(x_1,...,x_n;\ y_1,...,y_n;\ \sum_{k=1}^{m}s_{11}^{(k)}R_{11}^{(k)}(z_1),...,\sum_{k=1}^{m}s_{1n}^{(k)}R_{1n}^{(k)}(z_n)),$$

$$...,f_n(x_1,...,x_n;\ y_1,...,y_n;\ \sum_{k=1}^{m}s_{n1}^{(k)}R_{n1}^{(k)}(z_1),...,\sum_{k=1}^{m}s_{nn}^{(k)}R_{nn}^{(k)}(z_n)))^T$$

has Property (LM). Then, while $\max\{p_1(t),...,p_n(t)\}>0$ is nondecreasing on $(-\infty,t_0]$, and $||\psi||\equiv\max_{1\le i\le n}\{\sup_{-\infty<t\le t_0}p_i(t)\}$ is small enough, there exist a time $\bar{t}>t_0$ and a positive constant $\bar{M}$ which are independent of $||\psi||$ such that

$$p_1(\bar{t})+...+p_n(\bar{t})\ge\bar{M}.$$

If, in addition, $q=\sigma_0=\sigma_1=\sigma_2=+\infty$ and $F(x,y,z)$ has Property (M), then

$$\limsup_{t\to+\infty}(p_1(t)+...+p_n(t))=+\infty.$$

Remark 2. As shown in [20], the functions f_i and $A_{ij}(t,u)$ $(i = 1,2,...,n)$ satisfying the assumptioms of Lemma 1 are rather general. For example, while $f_i(i = 1,2,...,n)$ satisfy the following nonlinear inequality:

$$f_i \;\geq\; a_i p_i^{\alpha_i}(t) - \sum_{j=1}^{n}(b_{ij}\bar{p}_j^{\beta_{ij}}(t) + \int_{\theta}^{t} A_{ij}(t-u)p_j^{\gamma_{ij}}(u)du), \tag{3}$$

for $i = 1,2,...,n$, where $a_i > 0$, $b_{ij} \geq 0$, $\alpha_i > 0$, $\beta_{ij} > 0$ and $\gamma_{ij} > 0$ are constants; $A_{ij}(u)$ is a continuous nonnegative function for $i,j = 1,2,...,n$, it easily follows from Definitions 1 and 2 that the assumptions (iv) and (v) of Lemma 1 can be satisfied if the following conditions hold:

(i') $\alpha_i \leq \min_{1\leq j\leq n}\{\beta_{ij},\ \gamma_{ij}\}$;

(ii') $\int_0^{+\infty} A_{ij}(u)du \leq s_{ij} = \text{const.}$;

(iii') there exists a group of positive constants $d_1,...,d_n$ such that

$$a_i d_i^{\alpha_i} - \sum_{j=1}^{n}(b_{ij}\delta_{ij} + s_{ij}\tilde{\delta}_{ij})d_j^{\alpha_i} > 0,$$

where

$$\delta_{ij}\ (\tilde{\delta}_{ij}) = \begin{cases} 1 & \text{if} \quad \alpha_i = \beta_{ij} \quad (\alpha_i = \gamma_{ij}) \\ 0 & \text{if} \quad \alpha_i < \beta_{ij} \quad (\alpha_i < \gamma_{ij}) \end{cases},$$

for $i,j = 1,2,...,n$. Further, if the assumption (i') is replaced by the following stronger condition (i''):

(i'') $\alpha_0 \equiv \max_{1\leq i\leq n}\{\alpha_i\} \leq \min_{1\leq i,j\leq n}\{\beta_{ij},\ \gamma_{ij}\}$,

then it follows from Definition 1 that the above condition (iii') can be replaced with the following more practical condition (iii''):

(iii'') The matrix $D - (B + S)$ is an M-matrix, where

$$D = \text{diag}(a_1,...,a_n), \quad B = (b_{ij}\eta_{ij})_{n\times n}, \quad S = (s_{ij}\bar{\eta}_{ij})_{n\times n},$$

$$\eta_{ij}\ (\bar{\eta}_{ij}) = \begin{cases} 1 & \text{if} \quad \alpha_0 = \beta_{ij} \quad (\alpha_0 = \gamma_{ij}) \\ 0 & \text{if} \quad \alpha_0 < \beta_{ij} \quad (\alpha_0 < \gamma_{ij}) \end{cases}, \quad i,j = 1,2,...,n.$$

2. Instability Analysis on Neutral Nonlinear Differential Difference Systems with Infinite Delays

In this section, we will apply the inequality of the preceding section, together with the method of Liapunov functions, to the instability analysis of a class of nonlinear neutral differential difference systems with infinite delays and present a *easily verifiable* sufficient criterion. For differential difference systems with infinite delays, there exist some well developed fundamental theories. For example, for the case of retarded type, we refer to [5,6,11,23] and the Lecture Notes [12]; for the case of neutral type, we refer to [15,19,26,29]. In fact, [15,19,26] also contain excellent works with respect to boundedness, stability and periodic solutions etc. of neutral differential difference equations with unbounded and infinite delays.

Let C^n denote the space $C^n((-\infty,0],R^n)$ consisting of the real continuous functions mapping the interval $(-\infty,0]$ into R^n.

The neutral nonlinear differential difference systems considered in this paper are assumed to be of the following form,

$$\frac{d}{dt}Z(t,.) = H(t,Z(t,.)) + F(t,x(t),x(t-\Delta(t)),x_t), \tag{4}$$

where $Z(t,.)$ is a difference operator of the form

$$Z(t,.) = x(t) - D(t,x(t),x(t-\Delta(t)),x_t), \tag{5}$$

$x \in R^n$, $x_t = x(t+s)(-\infty \le \theta \le s \le 0)$; $H(t,x) : R_+ \times R^n \to R^n$ is a continuous function; $D(t,x,y,\phi)$, $F(t,x,y,\phi) : R_+ \times R^n \times R^n \times C^n \to R^n$ are continuous functionals with respect to their all arguments such that

$$H(t,0) = D(t,0,0,0) = F(t,0,0,0) = 0$$

for all $t \in R_+$; the delay function $\Delta(t) : R_+ \to R_+$ is continuous such that $t - \Delta(t) \to +\infty$ $(t \to +\infty)$.

Clearly, while $F(t,x,y,\phi) \equiv 0$ for all $(t,x,y,\phi) \in R_+ \times R^n \times R^n \times C^n$, system (4) is reduced to the following special form

$$\frac{d}{dt}Z(t,.) = H(t,Z(t,.)), \tag{6}$$

which is called a *completely integrable system* in [15]. The instability of the completely integrable system (6) and system (4) in general metric space M were considered in [14] and [15] by using the methods of Lyapunov functionals and the inversion theorem for Chetaev's theorem.

The initial condition of (4) is given as follows,

$$x(t_0+s) = \phi(s), \quad -\infty \le s \le 0, \tag{7}$$

where $t_0 \ge 0$ and $\phi \in BU \equiv \{\phi \mid \phi \in C^n$ is bounded and uniformly continuous on $(-\infty,0]\}$.

As usual, we say a continuous function $x(t)(t \in R)$ is the solution of (4) with the initial condition (7), if $Z(t,.) = x(t) - D(t,x(t),x(t-\Delta(t)),x_t)$ is continuously differentiable and satisfies (4) on $[t_0,+\infty)$ and $x(t)$ satisfies the initial condition (7). Clearly, (4) possesses the trivial solution $x(t) = 0$.

The main reasons for choosing the admissible Banach space BU with the uniform norm $||\phi|| \equiv \sup_{s\le 0} ||\phi(s)||$ for $\phi \in BU$ as the initial function space of (4) are that: (i) our purpose in this paper is to consider the instability of the trivial solution of (4); (*ii*) the fundamental theory of the initial problem (4) and (7) have been considered in [15], [19] and [29]; and (*iii*) the space BU can be included in some important phase spaces, for example, the admissible Banach spaces UC_g, C_γ and the Banach space BC (see [2,4,5,8,11-13,18] for details).

The instability of the trivial solution of (4) is defined as follows.

Definition 3. The trivial solution $x(t) = 0$ of (4) is said to be unstable, if there exists some constant $\bar{\epsilon} > 0$ such that for any small $\delta > 0$ and any $t_0 \geq 0$, there exist $\phi \in BU$ and $\bar{t} \geq t_0$ such that $||\phi|| \leq \delta$ and $||x(\bar{t},, t_0, \phi)|| \geq \bar{\epsilon}$.

We use the same symbol $||.||$ to denote the norms in R^n and BU, but no confusion will occur.

Let us list the following assumptions before we proceed further.

(A). For $t \geq 0$ and $||x(s)|| \leq h_1$ $(s \leq t,\ 0 < h_1 \leq +\infty)$,

$$||D(t, x(t), x(t-\Delta(t)), x_t)|| \leq \sum_{k=1}^{m} (c_k(t)||\bar{x}(t)||^{\beta_{1k}} + \int_\theta^t A_{1k}(t,u)||x(u)||^{\gamma_{1k}} du),$$

$$||F(t, x(t), x(t-\Delta(t)), x_t)|| \leq \sum_{k=1}^{m} (b_k(t)||\bar{x}(t)||^{\beta_{2k}} + \int_\theta^t A_{2k}(t,u)||x(u)||^{\gamma_{2k}} du),$$

where $||\bar{x}(t)|| = \sup_{-\Delta(t) \leq s \leq 0} ||x(t+s)||$; $b_k(t)$, $c_k(t)$, $A_{1k}(t,u)$ and $A_{2k}(t,u)$ are nonnegative continuous functions; β_{lk} and γ_{lk} are positive constants for $l = 1, 2$ and $k = 1, 2, ..., m$.

(B). There exists a continuous function $V(t,x) : R_+ \times R^n \to R$ such that for $t \geq 0$ and $||x|| \leq h_2$ $(0 < h_2 \leq +\infty)$,

$$(\alpha||x||)^{\theta_1} \leq V(t,x) \leq u(||x||), \quad ||(\frac{\partial V(t,x)}{\partial x})^T|| \leq q(t)||x||^{\theta_2} \quad \text{and}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} H(t,x) \geq r(t) V^{\theta_3}(t,x);$$

where $\alpha > 0$, $\theta_1 > 0$, $\theta_2 \geq 0$ and $\theta_3 > 0$ are constants such that $\theta_1\theta_3 - \theta_2 > 0$; the function $u(s) : [0, h_2) \to R_+$ is continuous and nondecreasing such that $u(s) > 0$ for $s > 0$ and $u(0) = 0$; the functions $r(t) : R_+ \to R_+$ and $q(t) : R_+ \to R_+$ are continuous such that for $t \geq 0$,

$$r(t) > 0 \quad \text{and} \quad \int_0^{+\infty} r(t)dt = +\infty.$$

Assume further that for all $t \geq 0$, any $s > 0$ and $k = 1, 2, ..., m$,

$$(i) \qquad \frac{q(t)b_k(t)}{r(t)} \leq \bar{b}_k = \text{const.}, \qquad c_k(t) \leq c_k = \text{const.},$$

$$\int_\theta^t A_{1k}(t,u)du \leq s_{1k} = \text{const.}, \quad \int_\theta^t \frac{q(t)A_{2k}(t,u)}{r(t)}du \leq \bar{s}_{2k} = \text{const.};$$

$$(ii) \qquad \lim_{t\to+\infty} \int_\theta^s (A_{1k}(t,u) + \frac{q(t)A_{2k}(t,u)}{r(t)})du = 0.$$

We are now in a position to state and prove our main result.

Theorem 1. *Assume that* $(A), (B), (i)$ *and* (ii) *hold, and*
(iii) $\theta_1\theta_3 - \theta_2 \leq \min\{\beta_{2k},\ \gamma_{2k}\}, \quad \beta_{1k} \geq 1, \quad \gamma_{1k} \geq 1;$
(iv) $\sum_{k=1}^m (c_k\delta_{1k} + s_{1k}\bar{\delta}_{1k}) < 1,$ *and*

$$\frac{\sum_{k=1}^m (\bar{b}_k\delta_{2k} + \bar{s}_{2k}\bar{\delta}_{2k})}{(1 - \sum_{k=1}^m (c_k\delta_{1k} + s_{1k}\bar{\delta}_{1k}))^{\theta_1\theta_3-\theta_2}} < \alpha^{\theta_1\theta_3},$$

where

$$\delta_{1k}\ (\bar{\delta}_{1k}) = \begin{cases} 1, & \text{if } \beta_{1k} = 1\ (\gamma_{1k} = 1) \\ 0, & \text{if } \beta_{1k} > 1\ (\gamma_{1k} > 1) \end{cases},$$

$$\delta_{2k}\ (\bar{\delta}_{2k}) = \begin{cases} 1, & \text{if } \theta_1\theta_3 - \theta_2 = \beta_{2k}\ (\theta_1\theta_3 - \theta_2 = \gamma_{2k}) \\ 0, & \text{if } \theta_1\theta_3 - \theta_2 < \beta_{2k}\ (\theta_1\theta_3 - \theta_2 < \gamma_{2k}) \end{cases},$$

for $k = 1, 2, ..., m.$ *Then,* $x(t) = 0$ *of* (4) *is unstable.*

Proof. Let us choose $q \in (0, +\infty]$ and the constant vector $\xi \in R^n$ satisfying

$$q \leq \min\{h_1^{\theta_1},\ (\alpha h_2)^{\theta_1}\}, \tag{8}$$

$$q^{\frac{1}{\theta_1}} + \sum_{k=1}^m (c_k q^{\frac{\beta_{1k}}{\theta_1}} + s_{1k} q^{\frac{\gamma_{1k}}{\theta_1}}) \leq h_2, \tag{9}$$

$$0 < ||\xi|| \leq \min\{h_1,\ q^{\frac{1}{\theta_1}}\}, \tag{10}$$

$$||\xi|| + \sum_{k=1}^m (c_k ||\xi||^{\beta_{1k}} + s_{1k}||\xi||^{\gamma_{1k}}) \leq h_2 \tag{11}$$

and

$$u(||\xi|| + \sum_{k=1}^m (c_k||\xi||^{\beta_{1k}} + s_{1k}||\xi||^{\gamma_{1k}})) \leq q. \tag{12}$$

For any $t_0 \geq 0$, let $x(t) = x(t, t_0, \xi)$ be the solution of (4) with the initial condition: $x(t_0 + s) = \xi\ (-\infty \leq s \leq 0)$.

In the following discussion, we shall show that the trivial solution of (4) is unstable by considering the above solution $x(t)$ with sufficiently small $||\xi||$.

Set

$$p_1(t) = \begin{cases} V(t, Z(t,.)), & t \geq t_0, \\ V(t_0, Z(t_0,.)), & t \leq t_0, \end{cases}, \quad p_2(t) = ||x(t)||^{\theta_1}, \quad t \in R. \tag{13}$$

From $(A), (B)$ and (13), it is easy to see that the functions $p_1(t)$ and $p_2(t)$ are continuous on R and that the function $\max\{p_1(t), p_2(t)\}$ is positive and nondecreasing on $(-\infty, t_0]$. Moreover, from $(A), (B)$ and $(10) - (13)$, we also have that $p_1(t)$ and $p_2(t)$ can be made arbitrarily small on $(-\infty, t_0]$ as long as $||\xi||$ is chosen small enough and that $p_l(t) \leq q$ $(t \leq t_0, \ l = 1, 2)$.

Since, in view of $(5), (8)$ and (9), $p_l(s) \leq q$ $(s \leq t, \ t \geq t_0, \ l = 1, 2)$ imply $||x(s)|| \leq h_1$ and $||Z(s,.)|| \leq h_2$ $(s \leq t, \ t \geq t_0)$, it follows from $(A), (B), (13)$ and (i) that for $t \geq t_0$ and $p_l(s) \leq q$ $(s \leq t, l = 1, 2)$,

$$\begin{aligned} D^+ p_1(t) &\geq r(t) V^{\theta_3}(t, Z(t,.)) + \frac{\partial V(t, Z(t,.))}{\partial Z} F(t, x(t), x(t - \Delta(t)), x_t) \\ &\geq r(t) V^{\theta_3}(t, Z(t,.)) - q(t) ||Z(t,.)||^{\theta_2} \sum_{k=1}^{m} (b_k(t) ||\bar{x}(t)||^{\beta_{2k}} \\ &\quad + \int_\theta^t A_{2k}(t, u) ||x(u)||^{\gamma_{2k}} du) \\ &\geq r(t) \{ p_1^{\theta_3}(t) - p_1^{\frac{\theta_2}{\theta_1}}(t) \alpha^{-\theta_2} \sum_{k=1}^{m} (\bar{b}_k \bar{p}_2^{\frac{\beta_{2k}}{\theta_1}}(t) \\ &\quad + \frac{q(t)}{r(t)} \int_\theta^t A_{2k}(t, u) p_2^{\frac{\gamma_{2k}}{\theta_1}}(u) du) \} \\ &= r(t) p_1^{\frac{\theta_2}{\theta_1}}(t) \{ p_1^{\frac{\theta_1 \theta_3 - \theta_2}{\theta_1}}(t) - \alpha^{-\theta_2} \sum_{k=1}^{m} (\bar{b}_k \bar{p}_2^{\frac{\beta_{2k}}{\theta_1}}(t) \\ &\quad + \frac{q(t)}{r(t)} \int_\theta^t A_{2k}(t, u) p_2^{\frac{\gamma_{2k}}{\theta_1}}(u) du) \} \\ &\equiv r(t) p_1^{\frac{\theta_2}{\theta_1}}(t) f_1(*), \end{aligned} \tag{14}$$

where $\bar{p}_l(t) = \sup_{-\Delta(t) \leq s \leq 0} p_l(t + s)$ for $l = 1, 2$.

On the other hand, again from $(A), (B), (5), (8), (9)$ and (i), we have for $t \geq t_0$ and $p_l(s) \leq q$ $(s \leq t, l = 1, 2)$,

$$\begin{aligned} 0 &\geq ||x(t)|| - ||Z(t,.)|| - ||D(t, x(t - \Delta(t)), x_t)|| \\ &\geq ||x(t)|| - ||Z(t,.)|| - \sum_{k=1}^{m} (c_k(t) ||\bar{x}(t)||^{\beta_{1k}} \\ &\quad + \int_\theta^t A_{1k}(t, u) ||x(u)||^{\gamma_{1k}} du) \end{aligned}$$

$$
\begin{aligned}
&\geq\ p_2^{\frac{1}{\theta_1}}(t) - \frac{1}{\alpha} p_1^{\frac{1}{\theta_1}}(t) - \sum_{k=1}^{m} (c_k \bar{p}_2^{\frac{\beta_{1k}}{\theta_1}}(t) \\
&\quad + \int_\theta^t A_{1k}(t,u) p_2^{\frac{\gamma_{1k}}{\theta_1}}(u) du) \\
&\equiv\ f_2(*).
\end{aligned} \tag{15}
$$

Clearly, from $(A), (B), (i)$ and (ii) of Theorem 1, it is easy to see that the inequalities (14) and (15) satisfy $(ii) - (iv)$ of Lemma 1 with $n = 2$, $k_1 = 1$, $k_2 = 0$, $r_1(t) = r(t)$, $r_2(t) = 1$, $b_1(u) = u^{\frac{\theta_2}{\theta_1}}$ and $b_2(u) = 1$. In the following, let us show that (14) and (15) also satisfy (v) of Lemma 1, i.e., the function $(f_1(*), f_2(*))^T$ has *Property* (LM).

In fact, $(f_1(*), f_2(*))^T$ has *Property* (LM) if and only if there exist two positive constants d_1 and d_2 such that for sufficiently small $u > 0$,

$$
(d_1 u)^{\frac{\theta_1\theta_3 - \theta_2}{\theta_1}} > \frac{1}{\alpha^{\theta_2}} \sum_{k=1}^{m} (\bar{b}_k (d_2 u)^{\frac{\beta_{2k}}{\theta_1}} + \bar{s}_{2k} (d_2 u)^{\frac{\gamma_{2k}}{\theta_1}})
$$

and

$$
(d_2 u)^{\frac{1}{\theta_1}} > \frac{1}{\alpha} (d_1 u)^{\frac{1}{\theta_1}} + \sum_{k=1}^{m} (c_k (d_2 u)^{\frac{\beta_{1k}}{\theta_1}} + s_{1k} (d_2 u)^{\frac{\gamma_{1k}}{\theta_1}}).
$$

By (iii) of Theorem 1, the above is clearly equivalent to

$$
(d_1)^{\frac{\theta_1\theta_3 - \theta_2}{\theta_1}} > \frac{1}{\alpha^{\theta_2}} \sum_{k=1}^{m} (\bar{b}_k \delta_{2k} + \bar{s}_{2k} \bar{\delta}_{2k}) (d_2)^{\frac{\theta_1\theta_3 - \theta_2}{\theta_1}}
$$

and

$$
(d_1)^{\frac{1}{\theta_1}} < \alpha \{1 - \sum_{k=1}^{m} (c_k \delta_{1k} + s_{1k} \bar{\delta}_{1k})\} (d_2)^{\frac{1}{\theta_1}},
$$

which are clearly equivalent to (iv) of Theorem 1.

Therefore, from Lemma 1, there exist a time $\bar{t} > t_0$ and a positive constant $\bar{M}$ which are indepentent of the initial vector ξ such that

$$
p_1(\bar{t}) + p_2(\bar{t}) \geq \bar{M}. \tag{16}
$$

We claim that (16) implies the trivial solution of (4) is unstable. If not, for any sufficiently small positive constant $\varepsilon \leq 1$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $||\xi|| \leq \delta$ implies $||x(t)|| \leq \varepsilon$ for $t \geq t_0$. Let ε be small enough such that

$$
\varepsilon(1 + \sum_{k=1}^{m} (c_k + s_{1k})) \leq h_2, \quad \text{and}
$$

$$
u(\varepsilon(1 + \sum_{k=1}^{m} (c_k + s_{1k}))) + \varepsilon^{\theta_1} < \bar{M}.
$$

Thus, from $(i), (iii), (13), (A)$ and (B), we have for $t \geq t_0$,

$$\begin{aligned} p_1(t) + p_2(t) &= V(t, Z(t,.)) + ||x(t)||^{\theta_1} \\ &\leq u(||Z(t,.)||) + ||x(t)||^{\theta_1} \\ &\leq u(\varepsilon(1 + \sum_{k=1}^{m}(c_k + s_{1k}))) + \varepsilon^{\theta_1} \\ &< \bar{M}, \end{aligned}$$

which contradicts to (16). This completes the proof of Theorem 1.

To illustrate the application of the preceding theorem, let us consider the neutral nonlinear scalar integro-differential equation

$$\frac{d}{dt}(x(t) - c(t)x^{\beta_1}(t - \Delta(t)) - \int_\theta^t k(t,s)x^{\gamma_1}(s)ds) = a(t)x^\nu(t) +$$
$$+b(t)x^{\beta_2}(t - \Delta(t)) + \int_\theta^t (r(t,s)x^{\gamma_2}(s) + p(t,s)x^\nu(s))ds, \tag{17}$$

where $x \in R$; ν, β_k and γ_k are positive constants; θ and $\Delta(t)$ are defined as in system (4); $a(t)$, $b(t)$, $c(t)$, $k(t,s), r(t,s)$ and $p(t,s)$ are scalar continuous functions for $t \geq 0$ and $\theta \leq s \leq t$.

Let $q(t,s)$ be a continuously differentiable function satisfying

$$\frac{\partial q(t,s)}{\partial t} = p(t,s), \qquad \theta \leq s \leq t, \tag{18}$$

then, (17) can be written as the following form:

$$\frac{d}{dt}(x(t) - c(t)x^{\beta_1}(t - \Delta(t)) - \int_\theta^t (k(t,s)x^{\gamma_1}(s) + q(t,s)x^\nu(s))ds)$$
$$= g(t)x^\nu(t) + b(t)x^{\beta_2}(t - \Delta(t)) + \int_\theta^t r(t,s)x^{\gamma_2}(s)ds, \tag{19}$$

where $g(t) = a(t) - q(t,t)$.

The above condition (18) was first introduced by Burton (see [4]), which shows that the function $a(t)$ can be vanished at any $t \geq 0$.

Systems (17) and (19) cover a very extensive class of nonlinear neutral integro-differential equations. For example, while $b(t) = c(t) = k(t,s) = r(t,s) = 0$ $(0 \leq s \leq t)$ and $\nu = 1$, (17) is reduced to well known linear retarded Volterra integro-differential system whose stability and instability have been studied well (see [4]) based on the mothod of Liapunov functionals. On the other hand, (17) and (19) may include some important linear and nonlinear integro-differential systems considered in $[4, 7, 13 - 15, 18, 19, 26]$ as special cases.

Now, for the most general nonlinear case, let us apply Theorem 1 to investigate the instability of system (19) under the following assumptions:

$$(i) \qquad \text{for all } t \geq 0, \quad g(t) = a(t) - q(t,t) > 0, \qquad \int_{t_0}^{+\infty} g(t)dt = +\infty;$$

$$(ii) \qquad \text{for all } t \geq 0, \quad |c(t)| \leq c = \text{const.}, \quad \frac{|b(t)|}{g(t)} \leq b = \text{const.},$$

$$\int_{\theta}^{+\infty} |k(t,s)|ds \leq k = \text{const.}, \qquad \int_{\theta}^{+\infty} |q(t,s)|ds \leq q = \text{const.},$$

$$\int_{\theta}^{+\infty} \frac{|r(t,s)|}{g(t)} ds \leq r = \text{const.};$$

$$(iii) \qquad \text{for any } u > 0, \quad \lim_{t \to +\infty} \int_{\theta}^{u} (|k(t,s)| + |q(t,s)| + \frac{|r(t,s)|}{g(t)})ds = 0.$$

We first rewrite system (19) as the form of system (4),

$$\frac{d}{dt} Z(t,.) \;=\; g(t) Z^{\nu}(t,.) + F(t,.), \tag{20}$$

where $Z(t,.) = x(t) - D(t,.)$ and

$$D(t,.) = c(t)x^{\beta_1}(t - \Delta(t)) + \int_{\theta}^{t} (k(t,s)x^{\gamma_1}(s) + q(t,s)x^{\nu}(s))ds,$$

$$F(t,.) = b(t)x^{\beta_2}(t - \Delta(t)) + \int_{\theta}^{t} r(t,s)x^{\gamma_2}(s)ds$$
$$+ g(t)(x^{\nu}(t) - (x(t) - D(t,.))^{\nu}).$$

Clearly,

$$|D(t,.)| \leq |c(t)||\bar{x}(t)|^{\beta_1} + \int_{\theta}^{t} (|k(t,s)||x(s)|^{\gamma_1} + |q(t,s)||x(s)|^{\nu})ds,$$

where $|\bar{x}(t)| = \sup_{-\Delta(t) \leq s \leq 0} |x(t+s)|$. Furthermore, if $\nu \geq 1$, then, from (ii), we easily have for $|x(s)| \leq h$ $(s \leq t,\ 0 < h < +\infty)$,

$$|x^{\nu}(t) - (x(t) - D(t,.))^{\nu}| \leq N(\nu, h)|D(t,.)|,$$

where $N(\nu, h) = \nu(h + ch^{\beta_1} + kh^{\gamma_1} + qh^{\nu})^{\nu - 1}$. Thus,

$$|F(t,.)| \leq |b(t)||\bar{x}(t)|^{\beta_2} + N(\nu, h)g(t)|c(t)||\bar{x}(t)|^{\beta_1}$$

$$+ \int_{\theta}^{t} (|r(t,s)||x(s)|^{\gamma_2} + N(\nu, h)g(t)(|k(t,s)||x(s)|^{\gamma_1} + |q(t,s)||x(s)|^{\nu}))ds.$$

Therefore, the functionals $D(t,.)$ and $F(t,.)$ satisfy the estimations in (A) with $m = 3$.

Now, define the Liapunov function $V(t,x)$ in (B) as $V(t,x) = x^2$, then, it is easy to see that, while ν can be written as the ratio of odd integers, (B) is also valid with $\theta_1 = 2$, $\theta_2 = 1$, $\theta_3 = \frac{1+\nu}{2}$, $\alpha = 1$, $q(t) = 2$ and $r(t) = 2g(t)$.

Observe that for $\nu > 1$, $N(\nu,h) \to 0 (h \to 0)$ and for $\nu = 1, N(\nu,h) = 1$, hence, from Theorem 1 we have

Proposition 1. *In addition to $(i) - (iii)$, assume further that:*
(iv) $\quad$ *ν is the ratio of odd integers, and*

$$1 \le \nu \le \min\{\beta_1,\ \beta_2, \gamma_1,\ \gamma_2\};$$

$(v)_1$ $\quad$ *for $\nu = 1$, $b\delta_2 + r\bar{\delta}_2 + 2(c\delta_1 + k\bar{\delta}_1 + q) < 1$;*
$(v)_2$ $\quad$ *for $\nu > 1$, $b\delta_2 + r\bar{\delta}_2 < 1$,*
where

$$\delta_1\ (\bar{\delta}_1) = \begin{cases} 1, & \text{if}\quad \beta_1 = 1\ (\gamma_1 = 1) \\ 0, & \text{if}\quad \beta_1 > 1\ (\gamma_1 > 1) \end{cases},$$

$$\delta_2\ (\bar{\delta}_2) = \begin{cases} 1, & \text{if}\quad \beta_2 = \nu\ (\gamma_2 = \nu) \\ 0, & \text{if}\quad \beta_2 > \nu\ (\gamma_2 > \nu) \end{cases}.$$

Then, the trivial solution of (19) *is unstable.*

Remark 3. If $p(t,s) = q(t,s) = 0$ for any $\theta \le s \le t$, the condition (iv) of Proposition 2 can be replaced with the following weaker the condition (iv'):
(iv') $\quad$ ν is the ratio of odd integers, and

$$0 < \nu \le \min\{\beta_2,\ \gamma_2\}, \quad \beta_1 \ge 1, \quad \gamma_1 \ge 1.$$

Remark 4. As system (17) is reduced to the systems considered in [4,7,13-15,18,19,26], the instability conditions given in Proposition 2 have symmetry with the stability conditions given in there.

Remark 5. Clearly, when the dimension of (4) is very high, as done in [17,20,21,24-26,28], we can further extend the preceding analysis techniques to the instability analysis of the large scale systems of (4).

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